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Critical Exponents of Manhattan Hamiltonian Walks in Two Dimensions, from Potts and $O(n)$ Models

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We consider a set of Hamiltonian circuits filling a Manhattan lattice, i.e., a square lattice with alternating traffic regulation. We show that the generating function (with fugacity z) of this set is identical to the critical partition function of a q-state Potts model on an unoriented square lattice with $q^{1/2}=z$. The set of critical exponents governing correlations of Hamiltonian circuits is derived using a Coulomb gas technique. These exponents are also found to be those of an $O(n)$ vector model in the low-temperature phase with $n=q^{1/2}=z$. The critical exponents in the limit $z=0$ are then those of spanning trees ($q=0$) and of dense polymers ($n=0$, $T<T_c$), corresponding to a conformal theory with central charge $C=-2$. This shows that the Manhattan orientation and the Hamiltonian constraint of filling all the lattice are irrelevant for the infrared critical properties of Hamiltonian walks.

KEY WORDS: Manhattan, Hamiltonian walk; critical exponents; Potts; $O(n)$; SOS; Coulomb gas; conformal invariance; surface exponents.

1. INTRODUCTION

A Hamiltonian walk is a self-avoiding walk that visits each site of a given lattice, thus filling completely the available space. A famous solution by Kasteleyn[1] gives the exact number of such Hamiltonian walks on any oriented lattice that is the covering graph of another oriented lattice, the enumeration reducing then to counting spanning trees of the underlying lattice, which in turn is obtained by evaluating a determinant.[1] A particular case of such a covering lattice is the Manhattan oriented square lattice, where successive horizontal (and vertical) lines are oriented in alternating directions.[1,2] The general counting of Hamiltonian walks on

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any lattice is not yet known (see Ref. 3 for a general discussion). The determination of the effective connectivity constant for Hamiltonian walks, in particular, has been the subject of recent research.[3-6]

Obviously, Hamiltonian walk problems have connections with the statistics of polymer melts or collapsed polymers.[7,8] Recently, progress has been made[9-11] for *dense* polymers in two dimensions, which are described by the low-temperature phase of the $O(n)$ n-vector model, in the limit $n=0$. By use of results of a standard Coulomb gas mapping,[12,13] an infinite series of exact critical exponents has been derived,[9] which can be used for enumerating the configurations of any dense, branched polymer with a specified topology.[14,9,11] These dense polymers occupy only a finite fraction of the lattice and are not Hamiltonian walks. However, one notes,[9,10] using the result of Kasteleyn,[1] that Hamiltonian walks on a Manhattan lattice correspond to a conformal theory with central charge $C=-2$ equal to that of the $O(n)$ n-vector model in its *low-temperature phase* for $n=0$. This suggests that Hamiltonian walks, even on oriented lattices, and dense polymers could be in the same universality class. Here we show that this is indeed the case. To prove it, we consider a Manhattan lattice, and generalize the original single Hamiltonian walk of Kasteleyn to a grand canonical set of disconnected Hamiltonian circuits filling the Manhattan lattice, with fugacity z. We show that the partition function (with free boundary conditions) can be transformed into that of a q-state Potts model with $q^{1/2}=z$, at the Potts critical point. An infinite set of critical exponents governing the correlation functions of Hamiltonian circuits with fugacity z, corresponding to a set of topologies of the circuits, is thus obtained.

We show that these exponents are also exactly those of an n-vector model in the *low-temperature phase*, with $z=q^{1/2}=n$. In the low-fugacity limit $z=n\to 0$ (a finite number of Hamiltonian circuits), one thus obtains the critical exponents of the $n=0$ model below T_c, i.e., dense polymers,[9,10] showing directly the identity of universality classes.

2. MANHATTAN HAMILTONIAN WALKS

Consider a Manhattan oriented lattice $\mathscr{M}$ (Fig. 1). It is the covering graph of an *oriented underlying square lattice* $\bar{\mathscr{S}}$, which is here in diagonal position (Fig. 1). The edges of the Manhattan lattice are obtained by joining the median points of the edges of $\bar{\mathscr{S}}$ and orienting them according to the orientation of $\bar{\mathscr{S}}$. Now, as remarked by Kasteleyn,[1] there is a one-to-one correspondence between an oriented Hamiltonian path on $\mathscr{M}$ and an oriented Eulerian path on $\bar{\mathscr{S}}$, i.e., a path that visits each edge of $\bar{\mathscr{S}}$ once and only once (Fig. 1). We take here *free* boundary conditions, and the possibility of having Hamiltonian *circuits* on a Manhattan lattice of $M\times N$

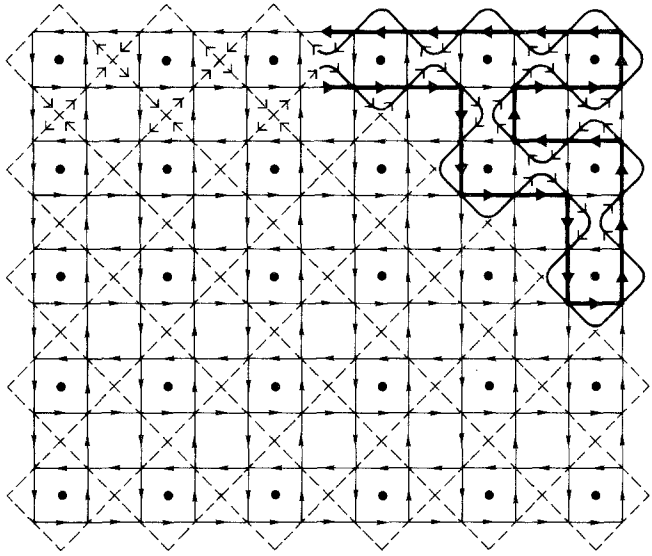

Fig. 1. The square oriented lattice $\mathscr{P}$ (diagonal dashed lines) and its covering graph, the square Manhattan oriented lattice $\mathscr{M}$ (horizontal and vertical solid lines). To any oriented Hamiltonian walk on $\mathscr{M}$ (on the upper right) there corresponds an oriented Eulerian walk on $\mathscr{P}$. These walks encircle points of an other (unoriented) square lattice $\mathscr{L}$ (dots) obtained by joining the centers of the counterclockwise plaquettes of $\mathscr{M}$.

sites requires then that M and N are even. Let us introduce another square lattice $\mathscr{L}$.[1] If we put a point in the center of each of those squares of lattice $\mathscr{M}$ whose sides are *counterclockwise* oriented, and if we connect all pairs of such points that are nearest neighbors by a line, we get[1] the *unoriented* quadratic lattice $\mathscr{L}$ (Fig. 1). A similar translated lattice $\mathscr{D}$ is associated with the squares with clockwise orientation[1] (Fig. 1), which plays a role for periodic boundary conditions.[1] It is the *dual* lattice of $\mathscr{L}$. Following Kasteleyn, one sees that a Hamiltonian circuit on $\mathscr{M}$ encircles, in a one-to-one correspondence, a graph on $\mathscr{L}$ (Fig. 1).

Now let us consider a set of oriented Hamiltonian circuits that fill completely the Manhattan lattice $\mathscr{M}$ (Fig. 2). We introduce a Hamiltonian grand canonical partition function with fugacity z

$$Z_{\mathrm{H}}(z) = \sum_{\mathscr{G} \in \mathscr{M}} z^{\mathscr{N}_P} \tag{1a}$$

where the sum is taken over all the possible graphs $\mathscr{G}$ made of $\mathscr{N}_P$ oriented Hamiltonian circuits filling $\mathscr{M}$, with $\mathscr{N}_P$ being summed over. One has also

$$Z_{\mathrm{H}}(z) = \sum_{K \geqslant 1} z^K N_{H,K} \tag{1b}$$

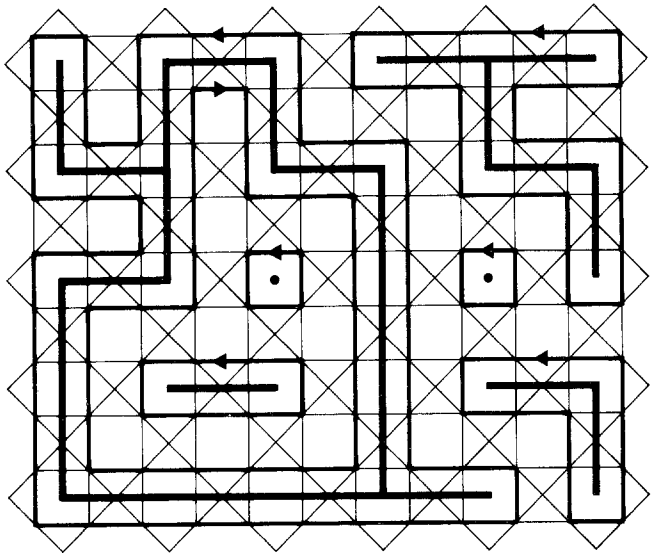

Fig. 2. A set of Hamiltonian circuits filling $\mathscr{M}$. The skeleton of the circuits spans graphs $\mathscr{G}'$ on $\mathscr{L}$, including isolated points, that cover all sites of $\mathscr{L}$. The diagonal unoriented square lattice is the surrounding lattice $\mathscr{S}$ of $\mathscr{L}$.

where $N_{H,K}$ is the total number of configurations of Hamiltonian circuits with K connected components. Note that here two Hamiltonian circuits can be inside each other (Fig. 2). As we shall see, the enumeration of $N_{H,K}$ is a solvable problem. (Another interesting problem consists in counting only K Hamiltonian circuits that are *adjacent* but never inside each other. This case has also just been solved[15] and its correlation functions exactly calculated.[15]). In Eq. (1) we have introduced a fugacity z for Hamiltonian loops. It must be noted that the standard meaning of Hamiltonian walks corresponds to a *finite* number of walks that fill the lattice, even in the large-lattice limit. In this limit, if $z > 0$, the average number $\langle \mathscr{N}_P \rangle$ of loops will become infinite with lattice size. The real Hamiltonian case ($\langle \mathscr{N}_P \rangle$ finite) will thus correspond to the limit $z \to 0$, to be taken at the end, in a way quite similar to the usual $n = 0$ limit of the $O(n)$ model for polymers.

It is straightforward to check[1] that each Hamiltonian circuit configuration $\mathscr{G}$ on $\mathscr{M}$ encircles an *unoriented spanning graph* $\mathscr{G}'$ on $\mathscr{L}$ in a one-to-one correspondence (Fig. 2). Hence, we may write

$$Z_{\mathrm{H}}(z) = \sum_{\substack{\text{spanning} \\ \text{graphs } \mathscr{G}' \in \mathscr{L}}} z^{\mathscr{N}_P} \tag{2}$$

$\mathscr{N}_P$ was the number of Hamiltonian circuits (or polygons) of graph $\mathscr{G}$ on $\mathscr{M}$. On the square lattice $\mathscr{L}$, $\mathscr{N}_P$ can be written in terms of the total

numbers $\mathcal{N}_L$ of loops and $\mathcal{N}_C$ of connected components (including isolated points) of the associated spanning graph $\mathcal{G}'$ as

$$\mathcal{N}_P = \mathcal{N}_L + \mathcal{N}_C \tag{3}$$

a relation that will be useful later.

3. POTTS MODEL

Spanning graphs on $\mathcal{L}$ (Fig. 2) are very reminiscent of the high-temperature expansion of the Potts model.[16] Consider then the q-state Potts model with Hamiltonian $\beta H = -\beta \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j}$, where $\sigma_i = 1,..., q$ (integer) and $\langle i, j \rangle$ denotes nearest neighbors on the unoriented square lattice $\mathcal{L}$. The high-temperature expansion of the partition function is then given by the Whitney polynomial[16,17]:

$$Z_{\text{Potts}}(q) = \sum_{\{\sigma\}} e^{-\beta H} = \sum_{\mathcal{G}'} W(\mathcal{G}') = \sum_{\mathcal{G}'} (e^{\beta} - 1)^{\mathcal{N}_B} q^{\mathcal{N}_C} \tag{4}$$

$W(\mathcal{G}')$ is the weight of a graph made of a total number $\mathcal{N}_B$ of bonds and $\mathcal{N}_C$ of connected components, including isolated points. Equation (4) now defines a model for any real or complex q. Now, a spanning graph $\mathcal{G}'$ on $\mathcal{L}$ can be associated with a polygon decomposition[17] of the *surrounding lattice* $\mathcal{S}$ of $\mathcal{L}$ (Figs. 2 and 3). Before being decomposed, the surrounding

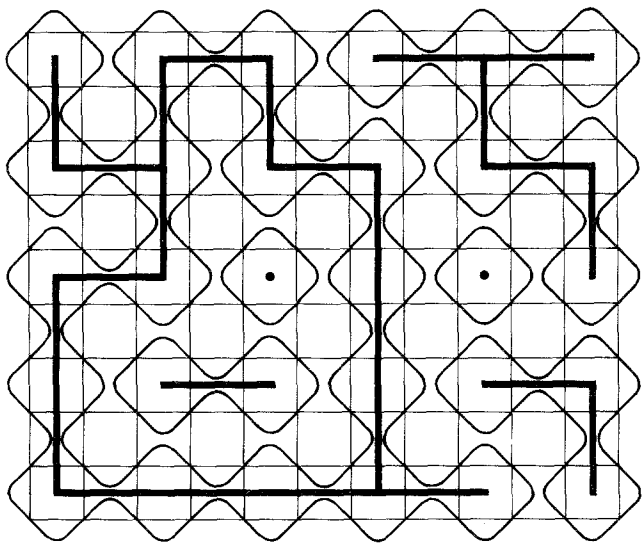

Fig. 3. The same spanning graph $\mathcal{G}'$ on $\mathcal{L}$ and the associated polygon decomposition of the surrounding lattice $\mathcal{S}$ of $\mathcal{L}$, where $\mathcal{S}$ is the original diagonal lattice $\bar{\mathcal{S}}$ without the orientation.

lattice $\mathscr{S}$ of a lattice $\mathscr{L}$ is defined as follows.[17] Draw simple polygons (here *diagonal squares*) surrounding each site of $\mathscr{L}$ such that: no polygons overlap, and no polygon surrounds another; polygons of nonadjacent sites of $\mathscr{L}$ have no common corner; polygons of adjacent sites of $\mathscr{L}$ have one and only one common corner. This corner is on (the middle of) the edge joining the sites. The corners of the polygons are the sites of $\mathscr{S}$. Note that here the surrounding lattice $\mathscr{S}$ is simply the diagonal square lattice and coincides with the former underlying square lattice $\bar{\mathscr{S}}$ of the Manhattan lattice $\mathscr{M}$, except that $\mathscr{S}$ is *unoriented* (Figs. 1–3), while $\bar{\mathscr{S}}$ is oriented. Now, in the *polygon decomposition* of the surrounding lattice $\mathscr{S}$, the rule is that some vertices of the surrounding lattice $\mathscr{S}$ are cut open to let the bonds of $\mathscr{L}$ go through unintersected. This also applies to the edges of the dual lattice $\mathscr{D}$ of $\mathscr{L}$. Note that the resulting polygon configuration appears as made of islands supporting the connected pieces of $\mathscr{G}'$ in lakes where all the sites of the dual lattice $\mathscr{D}$ of $\mathscr{L}$ are immersed.

On the square lattice $\mathscr{L}$, having a total number of sites $\mathscr{N}_S = MN/4$, one has (Euler's relation)

$$\mathscr{N}_L = \mathscr{N}_B + \mathscr{N}_C - \mathscr{N}_S \tag{5}$$

where, as in (3), $\mathscr{N}_L$ is the number of loops within the clusters of a spanning graph $\mathscr{G}'$. On the surrounding lattice $\mathscr{S}$, the total number of polygons one can draw around each cluster and in each loop (Fig. 3) is given by $\mathscr{N}_P = \mathscr{N}_L + \mathscr{N}_C$, and is clearly identical with the original number (3) of Hamiltonian circuits on $\mathscr{M}$ (compare Figs. 2 and 3).

Using Eqs. (3) and (5), one can rewrite $Z_{\text{Potts}}(q)$ in (4)

$$Z_{\text{Potts}}(q) = q^{\mathscr{N}_S/2} \sum_{\mathscr{G}'} [(e^{\beta} - 1)\, q^{-1/2}]^{\mathscr{N}_B}\, q^{\mathscr{N}_P/2} \tag{6}$$

In the infinite-lattice limit the Potts model is critical for $q \in [0, 4]$,[16] and its critical point is known by duality on a square lattice to be

$$(e^{\beta_C} - 1)\, q^{-1/2} = 1 \tag{6a}$$

Hence, at criticality, (6) reduces to

$$Z_{\substack{\text{Potts} \\ \text{critical}}}(q) = q^{\mathscr{N}_S/2} \sum_{\mathscr{G}'} q^{\mathscr{N}_P/2} \tag{7}$$

Thus, comparing (2) and (7), we obtain

$$Z_{\text{H}}(z) = q^{\mathscr{N}_S/2} Z_{\substack{\text{Potts} \\ \text{critical}}}(q) \qquad \text{for} \quad z = q^{1/2} \tag{8}$$

Hence, Z_H and $Z_{\text{Potts critical}}$ are identical, up to a trivial global factor, and due to their forms (2) and (7), are the partition functions of a gas of dense polygons that do not overlap, but may be inside each other. The identity (8) then implies that Hamiltonian circuits (or walks) form a *critical* system in the thermodynamic limit. This is established here for a Manhattan lattice, but this holds true in general. Therefore, one expects for Hamiltonian walks the appearance of nontrivial critical exponents. They will be determined in the next section from those of the associated Potts model.

Let us give a first application. The critical free energy of the Potts model is exactly known on the square lattice[18] in the thermodynamic limit

$$f(q, \beta_C) = \lim_{\mathcal{N}_S \to \infty} \ln Z_{\substack{\text{Potts} \\ \text{critical}}}(q)/\mathcal{N}_S$$

$$= \frac{1}{2} \ln q + \int_{-\infty}^{+\infty} \frac{dx}{x} \operatorname{th} \mu x \frac{\operatorname{sh}(\pi - \mu)x}{\operatorname{sh} \pi x} \tag{9}$$

with, for $q \leqslant 4$, $\cos \mu = q^{1/2}/2$ and $\mu \in [0, \pi/2]$. The integral $\mathcal{J}$ in (9) is analytic in μ and its expansion reads, around $\mu = \pi/2$ ($q = 0$),

$$\mathcal{J} = I_0 + \left(\mu - \frac{\pi}{2}\right) I_1 + \left(\mu - \frac{\pi}{2}\right)^2 \frac{1}{2} I_2 + \cdots$$

with

$$I_0 = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{x} \frac{\operatorname{sh} x}{\operatorname{ch}^2 x} = \frac{4G}{\pi}$$

where G is Catalan's constant, $G = 1 - 3^{-2} + 5^{-2} + \cdots$, and

$$I_1 = -\frac{2}{\pi} \int_0^{\infty} dx \left(\frac{1}{\operatorname{ch} x} - \frac{1}{\operatorname{ch}^3 x}\right) = -\frac{1}{2}$$

$$I_2 = \left(\frac{2}{\pi}\right)^2 \int_0^{\infty} dx \frac{x}{\operatorname{sh} x} \left(1 - \frac{3}{\operatorname{ch}^2 x}\right) = -2 - \frac{6}{\pi}$$

The corresponding expansion in terms of $q^{1/2}$ is

$$\mathcal{J} = I_0 - \tfrac{1}{2} q^{1/2} I_1 + \tfrac{1}{8} q I_2 + \cdots$$

From (8) and (9), we find

$$\lim_{\mathcal{N}_S \to \infty} \ln Z_H(z)/\mathcal{N}_S = \mathcal{J}$$

Identifying the expansion of $\ln Z_{\mathrm{H}}(z)$ in (1b) with that of $\mathscr{J}$ in terms of $q^{1/2}=z$, one finds to first order

$$\lim_{\mathscr{N}_S\to\infty}\frac{1}{4\mathscr{N}_S}\ln N_{\mathrm{H},1}=\frac{G}{\pi} \tag{10}$$

where $4\mathscr{N}_S=MN$ is the number of sites of $\mathscr{M}$, and the new exact results

$$\lim_{\mathscr{N}_S\to\infty}\frac{1}{\mathscr{N}_S}\frac{N_{\mathrm{H},2}}{N_{\mathrm{H},1}}=\frac{1}{4}$$

$$\lim_{\mathscr{N}_S\to\infty}\frac{1}{\mathscr{N}_S}\left(\frac{N_{\mathrm{H},3}}{N_{\mathrm{H},1}}-\frac{1}{2}\frac{N_{\mathrm{H},2}^2}{N_{\mathrm{H},1}^2}\right)=-\frac{1}{4}-\frac{3}{4\pi}$$

$$\vdots$$

Equation (10) is just Kasteleyn's result,[1] derived here by a different method. Further expanding (9) gives in principle $N_{\mathrm{H},K}$ to any order.

4. SOS, COULOMB GAS MODELS, AND CRITICAL EXPONENTS

The polygon decomposition of the surrounding lattice $\mathscr{S}$ (Fig. 3) allows one to consider $Z_{\mathrm{Potts}}(q)$ in (6) and (7) as the partition function of a special kind of six-vertex model or solid-on-solid (SOS) model. We state the facts we need here and refer the reader to previous work[12,13,19–24] for more details. An orientation is assigned to the polygons, after which they can be interpreted as steps (of $\pm\pi/2$) in a configuration of an SOS model. (Note that this orientation on $\mathscr{S}$ will be summed over and is *not* related to the former orientation on the Manhattan lattice $\mathscr{M}$.) One introduces height variables θ_x at sites X of the lattice $\mathscr{L}$ and of the dual lattice $\mathscr{D}$, and an oriented polygon on a lattice $\mathscr{S}$ is considered as a wall between regions of constant height, the highest region being on the left of the arrows. The Boltzmann weight of an SOS configuration on $\mathscr{S}$ is simply the product of phase factors e^{iu} (e^{-iu}) for each left (right) turn at one corner of the walls. In fact, on a square lattice, the difference between the numbers of left and right turns along a polygon with no self-crossing is always ± 4, and since the polygon orientations in the SOS model are summed over, yielding a factor $2\cos 4u$ for each polygon, one has

$$Z_{\mathrm{SOS}}(u)=\sum_{\mathscr{P}'}(2\cos 4u)^{\mathscr{N}_P}$$

and (at criticality)

$$Z_{\substack{\mathrm{Potts}\\ \mathrm{critical}}}(q)=q^{\mathscr{N}_S/2}Z_{\mathrm{SOS}}(u)$$

provided $2\cos 4u = q^{1/2}$. Hence, we find also for Hamiltonian circuits $Z_H(z) = Z_{SOS}(u)$ for $2\cos 4u = z$. The SOS model is driven by renormalization onto a critical Coulomb gas[12,13] with a coupling constant g given by[12,13]

$$q = z^2 = 2 + 2\cos \tfrac{1}{2}\pi g, \qquad g \in [2, 4], \quad q \in [0, 4], \quad z \in [0, 2]$$
$$8u/\pi = |2 - g/2| \bmod 4 \tag{11}$$

This transformation of the Hamiltonian model or Potts model into six-vertex and SOS models also allows one to calculate critical exponents. Let us introduce correlation functions[25] for the Hamiltonian model or the Potts model

$$G_k(X - Y) = \frac{1}{Z_{\text{Potts}}(q)} \sum_{\mathcal{G}'_k} W(\mathcal{G}'_k) \tag{12}$$

where the weight $W(\mathcal{G}'_k)$ is defined in (4), and where the sum is taken over all graphs $\mathcal{G}'_k$ of the surrounding lattice $\mathcal{S}$ formed by k polygons that join a neighborhood of a point X to a neighborhood of a point Y (the case $k = 3$ is represented in Fig. 4). On the Manhattan lattice $\mathcal{M}$, which covers the surrounding lattice $\mathcal{S}$, G_k is also the correlation function between k Hamiltonian circuits (Fig. 5)

$$G_k(X - Y) = \frac{1}{Z_H(z)} \sum_{\mathcal{G}_k} z^{\mathcal{N}_P(\mathcal{G}_k)} \tag{13}$$

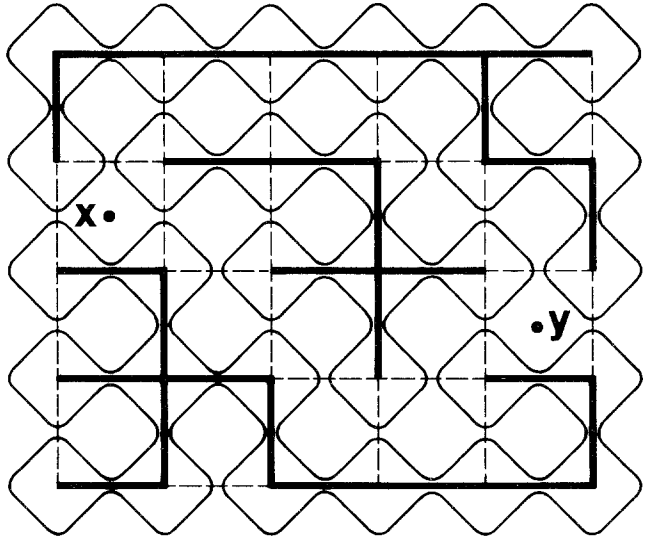

Fig. 4. Three polygons correlated at points X and Y, contributing to $G_3(X - Y)$ (here for $q = 0$).

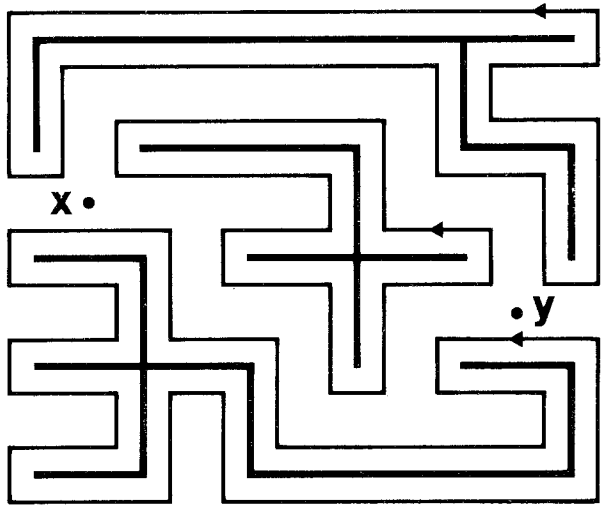

Fig. 5. The $k=3$ Hamiltonian circuits on $\mathcal{M}$ corresponding to Fig. 4 that come close together at X and Y and contribute to $G_3(X-Y)$. There are no other circuits on $\mathcal{M}$ ($z=0$). For $z\neq 0$, these three circuits are combined with loops.

where the sum is over graphs $\mathcal{G}_k$ of $\mathcal{M}$ formed by Hamiltonian circuits filling $\mathcal{M}$, among which (at least) k circuits join X to Y. The other circuits of $\mathcal{G}_k$ in general do not join X to Y, but some may occasionally. According to the identity (8), the Manhattan correlation function G_k in (13) is identical to the Potts one in (12) at the critical point (6a). The critical behavior of G_k will be $G_k(X-Y)=|X-Y|^{-2x_k}$, where the equality stands up to a factor, and where x_k are new universal scaling dimensions characteristic of Manhattan circuits and of the Potts model.

The critical behavior of G_k [Eq. (12)] of the Potts model has been recently calculated[25] for the purpose of studying percolation hull exponents for $q=1$. One uses the SOS model and the method follows and generalizes that introduced by den Nijs[23] for calculating chiral and cubic crossover exponents in the Potts model and by Nienhuis and Knops[24] for the Potts spinor exponents. In the SOS model, the polygons are arbitrarily oriented. For taking into account the correlation of k polygons between X and Y, one modifies the orientation of some parts of these polygons between X and Y in such a way that one circulates only from X to Y (Fig. 6).[23-25] Then X is a source of $2k$ lines and Y their sink. In terms of the heights of the SOS model, the resulting configuration $\mathcal{C}'_k$ corresponds to a screw dislocation between X and Y. Along a closed path around vortex X (Y), described on the trigonometric direction, the SOS height varies by $2k\times\pi/2=k\pi$ ($-k\pi$). These vortices correspond[23,24] in the

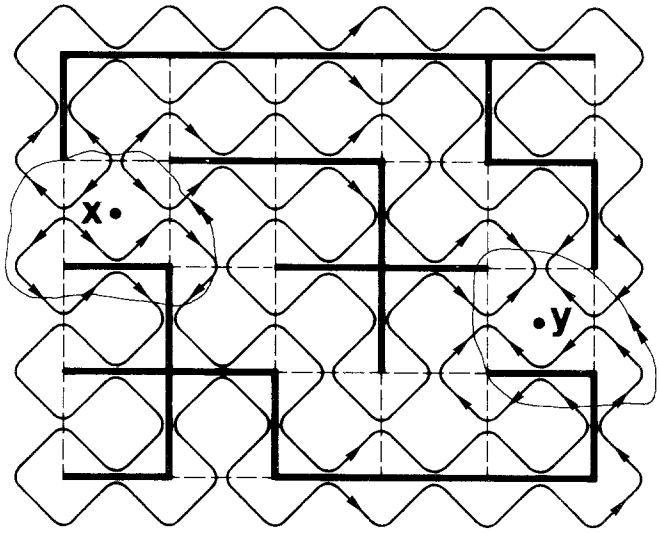

Fig. 6. The modification of orientation of the lines of the polygons joining X to Y. One creates a source at X and a sink at Y. Since $k=3$, one cannot accommodate on a discrete square lattice three polygons touching each other, so the rupture of orientation occurs only in neighboring points. When one circulates around X (Y) (double arrow) on a large enough path that encloses the orientation breaking points, the number of dislocation steps is $+6$ (-6).

Coulomb gas model to a magnetic charge or vorticity $m_X = -m_Y = k/2$. Next one has to take care of the proper phase rule in the SOS model. For instance,[24] if one winds the extremity of one of the polygons around Y, say, as represented in Fig. 7 (inspired by Ref. 24), the Boltzmann weight in the SOS model changes by a phase factor, since the two oriented sections of the polygon contribute oppositely to the winding number. For each left (right) full turn of the two lines one gets a factor e^{8iu} (or e^{-8iu}) in the SOS weight, with respect to the weight without winding. But since the left (right) rotation moves the height θ_Y at Y by $2 \times \pi/2 = \pi$ ($-\pi$), this phase factor can be repaired by multiplying the SOS weight $W_{\mathrm{SOS}}(\mathscr{C}'_k)$ by a spin-wave operator $\exp(ie_Y\theta_Y)$, where $e_Y = -8u/\pi$. A similar operator $\exp(ie_X\theta_X)$ has to be introduced at X, with $e_X = e_Y$. Hence we write[25]

$$G_k(X-Y) = [Z_{\substack{\text{Potts} \\ \text{critical}}}(q)]^{-1} \sum_{\mathscr{C}'_k} W_{\mathrm{SOS}}(\mathscr{C}'_k) \exp[i(e_X\theta_X + e_Y\theta_Y)]$$

where e_X and e_Y are two electric charges $e_X = e_Y = -8u/\pi = -|g/2 - 2|$ mod 4. The G_k appears[25] as the correlation function of two combinations of vortex and spin wave with respective magnetic and electric charges

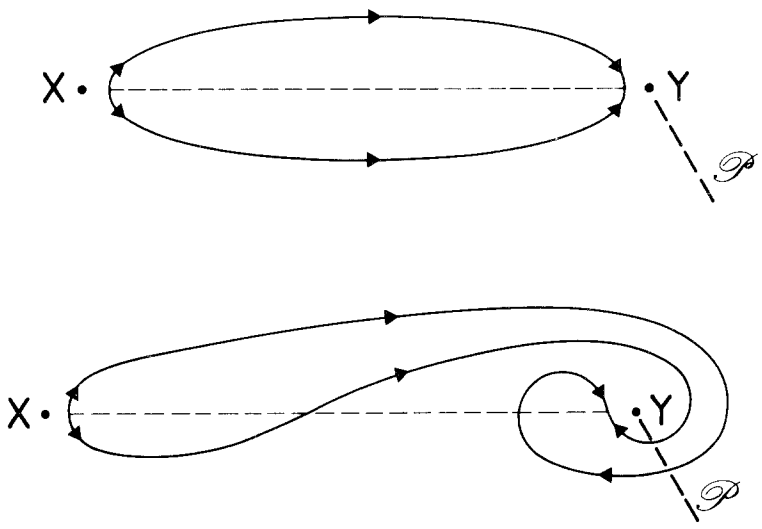

Fig. 7. A dislocation (dashed line) in the case of a $k=1$ polygon (in the continuum limit). When the polygon winds around Y, new curvature factors arise. Here, the upper arrow turns by -3π, the lower one by $-\pi$, resulting in -4π. The height at Y along a path $\mathscr{P}$ is lowered by $\Delta\theta_Y = -\pi$. The weight $W_{SOS} \exp[-i(8u/\pi)\,\theta_Y]$ is then left invariant.

$(m_X, e_X) = (k/2,\ g/2-2)$ and $(m_Y, e_Y) = (-k/2,\ g/2-2)$. It decays at criticality like

$$G_k(X-Y) = |X-Y|^{-2x_k} \tag{14}$$

with a critical exponent given by the den Nijs–Nienhuis Coulomb gas formula[12,13]

$$x_k = -\frac{g}{2} m_X m_Y - \frac{1}{2g} e_X e_Y = \frac{g}{2}\left(\frac{k}{2}\right)^2 - \frac{(4-g)^2}{8g} \tag{15}$$

This result[25] is valid for any value of q, $q \in [0, 4]$, with $g \in [2, 4]$, and thus gives the critical decay of the k-Hamiltonian circuit correlation functions (13) in the presence of a fugacity z, $z \in [0, 2]$, with the correspondence (11).

For $k=1$, G_1 corresponds to the correlation between two points of a single polygon or, on the Manhattan lattice, of a single Hamiltonian circuit immersed inside a grand canonical set of Hamiltonian circuits. Its critical exponent is then

$$x_1 = 1 - 2/g$$

Note that for a *true* Hamiltonian circuit filling the lattice, $q=0$, $z=0$, and $g=2$. Hence $x_1 = 0$. This is expected, since $G_1(X-Y)$ is the probability

that two points of the lattice belong to the same circuit, i.e., $G_1 = 1$, in the absence of any other loop.

Using standard scaling arguments, one can show that x_1 is related to the fractal dimension D of a large Hamiltonian circuit by[25]

$$D = 2 - x_1 = 1 + 2/g$$

This fractal dimension then varies from the upper bound $D = 2$ for $g = 2$, $q = 0$, $z = 0$ (true Hamiltonian circuit) to the lower bound $D = 3/2$ for $g = 4$, $q = 4$, $z = 2$ (XY model at the Kosterlitz–Thouless point). The fractal dimension $D = 2$ for a true Hamiltonian path is naturally required for a path filling a 2D lattice. It is also interesting to note that $q = 1$ ($g = 8/3$) is the percolation problem[26] and corresponds to $z = 1$ in the Hamiltonian problem. This reduces in (1) to a sum over all possible nonconnected closed paths visiting all sites of $\mathcal{M}$ once and only once. Then the fractal dimension of a typical Hamiltonian circuit is $D = 7/4$ and is identical to the hull fractal dimension of a percolation cluster, which has been recently derived.[25]

5. $O(n)$ MODEL

Let us show that the critical exponents (15) are equivalently those of the $O(n)$ model, $n \in [0, 2]$, in its low-temperature phase, which is also critical.[13] On a hexagonal lattice $\mathcal{H}$, an $O(n)$ model is defined by the partition function[13]

$$Z_{O(n)} = \int \prod_i d\mathbf{S}_i \prod_{\langle j,l \rangle} (1 + \beta \mathbf{S}_j \cdot \mathbf{S}_l)$$

where i, j, l are lattice sites, $\langle j, l \rangle$ denotes nearest neighbors, and $\mathbf{S}$ is an n-component vector with $|\mathbf{S}|^2 = n$. It is well known[27,28] that $Z_{O(n)}$ can be written as a sum over all diagrams consisting of closed and nonintersecting rings on the honeycomb lattice $\mathcal{H}$,

$$Z_{O(n)} = \sum_{\text{graphs}} \beta^{\mathcal{N}_B} n^{\mathcal{N}_P} \tag{16}$$

where $\mathcal{N}_B$ is the total number of bonds of the rings, i.e., their length, and $\mathcal{N}_P$ their number.

Now, comparing expansion (16) with the former expansions (1) and (2) of $Z_{\mathrm{H}}(z)$ and (7) of $Z_{\text{Potts critical}}(q)$ immediately suggests that these partition functions correspond in (16) to $\beta = 1$ and

$$n = q^{1/2} = z \tag{17}$$

As is intuitively obvious, the loop fugacity z in the Manhattan problem is also the loop fugacity n in the $O(n)$ model. Now, on a hexagonal lattice the critical point is known[13] exactly to be $\beta_C = [2+(2-n)^{1/2}]^{-1/2}$, $n \in [-2, 2]$. Hence $\beta_C < 1$ for any n and thus $\beta = 1$ in (16) corresponds to the critical *low-temperature phase* $\beta > \beta_C$ of the $O(n)$ model. One might argue that in (1) and (7) the Hamiltonian circuits or the polygons fill the lattice, while in (16) loops filling the lattice occur for $\beta \to \infty$, i.e., at $T = 0$. The whole low-temperature phase of the $O(n)$ model actually has the same critical properties.[13] In the critical domain the $O(n)$ model (16) for $n \in [-2, 2]$ can be transformed[13] into a Coulomb gas having a renormalized coupling constant g' such that

$$n = -2 \cos \pi g' \tag{18a}$$

The analytic determination of g' is

$$g' \in [1, 2] \tag{18b}$$

for the critical point $\beta = \beta_C$. According to Nienhuis,[13] the critical low-temperature phase corresponds to the other analytic determination

$$g' \in [0, 1] \tag{18c}$$

Due to (11) and (18) we see that Eq. (17) holds for

$$g' = \tfrac{1}{4} g \tag{19}$$

Hence the Potts critical model with $g \in [2, 4]$ and $q \in [0, 4]$ should correspond to the critical *low-temperature phase* of an $O(n)$ model with $g' \in [\frac{1}{2}, 1]$, $n \in [0, 2]$. (Note that a similar correspondence exists[29] between the Potts *tricritical* model for which $g \in [4, 6]$ and $q \in [0, 4]$ and the *critical* point of the $O(n)$ model $g' \in [1, 3/2]$ and $n \in [0, 2]$.)

Let us establish the identity of Potts and $O(n)$ critical exponents for (17) and (19). We consider the totally connected correlation function of the type

$$G_{O(n),k}(X - Y) = \langle \mathbf{S}_{i_1} \cdot \mathbf{S}_{i_2}, \ldots, \mathbf{S}_{i_{2k-1}} \cdot \mathbf{S}_{i_{2k}}, \mathbf{S}_{j_1} \cdot \mathbf{S}_{j_2}, \ldots, \mathbf{S}_{j_{2k-1}} \cdot \mathbf{S}_{j_{2k}} \rangle_{\text{total conn}} \tag{20}$$

where $i_1, \ldots, i_{2k}$ denote $2k$ points in the neighborhood of X, and $j_1, \ldots, j_{2k}$ denote $2k$ points in the neighborhood of Y; X and Y are mutually remote (Fig. 8). Since we want to reproduce a correlation function similar to the polygon correlation function G_k of (12) and (13), we select only diagrams in the high-temperature expansion of the $O(n)$ model where the $2k$ lines

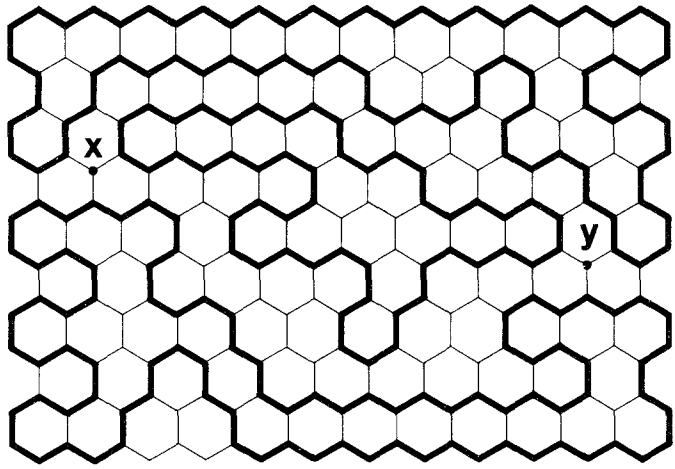

Fig. 8. A typical diagram contributing to $G_{O(n),k}(X-Y)$ (here for $n=0$ and $k=3$). The configuration has been chosen to be quite similar to that represented in Figs. 4 and 5, but the paths no longer fill the lattice completely, and the Manhattan orientation is lost. The universality class is nevertheless the same.

emerging from X actually go to Y and do not close onto themselves. [Thus, $G_{O(n),k}$ is not the simply connected correlation function of the $O(n)$ model.] Another way to obtain it would be to attribute different component indices to the $2k$ vectors $\mathbf{S}$ at X related in a one-to-one correspondence to the $2k$ vector indices at Y. The critical behavior of $G_{O(n),k}$ can then be calculated[13,30] by transforming[13] the $O(n)$ model into a particular SOS model (the triangular SOS model) and a Coulomb gas. Heights θ_X are attributed to the centers of the hexagons on the dual triangular lattice, which vary by steps of $\pm\pi$. Then *oriented* closed rings or polygons on the hexagonal lattice $\mathscr{H}$ represents domain walls between regions of constant heights, the higher one being on the left of each arrow. The TSOS weight is simply obtained as a product of factors βe^{iu} (βe^{-iu}) for each left (right) turn on the hexagonal lattice $\mathscr{H}$ resulting for a non-self-intersecting left (right) ring on $\mathscr{H}$ into a global term $\beta^{\mathscr{N}} e^{6iu}$ ($\beta^{\mathscr{N}} e^{-6iu}$), where $\mathscr{N}$ is the number of bonds of the ring. Then, by summing over independent orientations of the polygons, one sees[13] that $Z_{\mathrm{TSOS}} = Z_{O(n)}$ [Eq. (16)] for

$$n = 2\cos 6u, \qquad 6u/\pi = |1-g'| \bmod 2 \tag{21}$$

For calculating $G_{O(n),k}(X-Y)$ of (20), one proceeds[13,30] in a way similar to that used for the Potts model. One modifies in the TSOS model the orientation of some lines of the k polygons joining X to Y in such a way that one circulates only from X to Y. Then $(\mathbf{S}\cdot\mathbf{S}')^k(X)$ acts as a source of

$2k$ domain walls and Y as a sink. There is thus a dislocation between X and Y, with step $2k\pi$ ($-2k\pi$) around X (Y).

This corresponds[13,30] in the Coulomb gas to magnetic charges $m_X = -m_Y = k$. As above, new curvature effects appear[13] when a dislocation is present in the TSOS model. They are repaired[13,30] by a spin-wave operator $\exp[i(e_X\theta_X + e_Y\theta_Y)]$ with electric charge

$$e_X = e_Y = -(6u/\pi) \bmod 2$$

Taking the lowest electric charges in order to obtain the dominant critical behavior, one has

$$e_X = e_Y = 1 - g'$$

Hence $G_{O(n),k}$ has the critical behavior

$$G_{O(n),k} = |X - Y|^{-2x'_{2k}}$$

with

$$x'_{2k} = -\frac{g}{2} m_X m_Y - \frac{1}{2g} e_X e_Y = \frac{g'}{2} k^2 - \frac{1}{2g'}(1 - g')^2 \tag{22}$$

We check that for $g' = \frac{1}{4}g$, $q^{1/2} = n$, the Potts exponents x_k in (15) and the $O(n)$, $T < T_c$ exponents x'_{2k} in (22) are *identical.* Q.E.D.

It is also interesting to calculate the central charge C of the associated conformally invariant theory.[31] For the Potts and $O(n)$ models, we use the Coulomb gas representation[31,32]

$$C_q = 1 - 3(4 - g)^2/2g, \qquad C_{O(n)} = 1 - 6(1 - g')^2/g' \tag{23}$$

the latter being valid both for $\beta = \beta_C$ and $\beta > \beta_C$. From (19) we get, as expected, $C_q = C_{O(n)}$ for $q^{1/2} = n$. For the Hamiltonian fugacity $z = n \in [0, 2]$, $g' \in [\frac{1}{2}, 1]$, one thus has a central charge $C \in [-2, 1]$. The value $C = -2$ corresponds[9] to usual Hamiltonian walks and dense polymers, and $C = 1$ (free field) to the XY model ($q = 4$ Potts model).

The critical exponents x_k [Eq. (15)] $= x'_{2k}$ [Eq. (22)] also belong to the Kac table of the associated conformal theory.[33] With the usual parametrization[31]

$$h_{p,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \qquad C = 1 - \frac{6}{m(m+1)} \tag{24}$$

we find, using (15) or (22), and (23)

$$x_k = x'_{2k} = 2h_{0,k} \tag{25}$$

with the identification $m = g/(4 - g) = g'/(1 - g')$. Thus, the conformal

parameter m takes the values $m \in [1, \infty[$, $m=1$ corresponding to dense polymers and Hamiltonian walks, $m=\infty$ to the XY model.

6. $q, n=0$ LIMIT

Of special interest is the zero-fugacity limit $z, q, n \to 0$. It corresponds, on an infinite lattice, to a finite number of Hamiltonian paths filling the Manhattan lattice, i.e., to the *usual* Hamiltonian problem. The associated $q \to 0$ limit of the Potts model also describes spanning trees[34] and the Kirchhoff problem.[35] The $n \to 0$ limit (for $T < T_c$) describes *dense* polymers.[9,10] All these models are in the same universality class.

The correlation function $G_k(X-Y)$ in (13) then corresponds to k Manhattan Hamiltonian circuits connecting X to Y (Fig. 5). Since $z=0$, there are *no other circuits* on $\mathcal{M}$. Their internal skeletons thus build k spanning trees on the square lattice $\mathcal{L}$.

In the $z = q^{1/2} = n \to 0$ limit, the critical exponents (15) read

$$x_k = \tfrac{1}{4}(k^2 - 1) \tag{26}$$

and are the exact critical exponents of Hamiltonian circuits on a 2D Manhattan lattice. But they are also those obtained[9] for simple *dense* polymers from the $n=0$ vector model at $T < T_e$. We found the exponents

$$x'_L = (L^2 - 4)/16$$

governing the critical decay of the correlation function of L linear dense polymers tied together at their extremities. On a Manhattan lattice, k circuits joining X to Y (Fig. 5) are actually $L = 2k$ polymers between X, and Y, and indeed $x_k = x'_{2k}$. The Manhattan orientation only prevents the accommodation of an *odd* number of linear Hamiltonian walks.

The universality class of Hamiltonian walks and of dense polymers is thus unique. The constraint of completely filling a lattice, and here the special Manhattan orientation, are *irrelevant* in the infrared, i.e., the critical large-distance limit.

We have shown here the universality of exponent x_k. From these basic scaling dimensions, one can calculate other critical exponents, such as the γ exponent of any branched polymer with fixed topology[9,14] and contact exponents[36] of a (dense or dilute) SAW. Hence the universality of Hamiltonian walks will also hold for these exponents.

7. SURFACE EXPONENTS

Besides bulk critical exponents, there are also surface critical exponents,[37,38] which play a role when one looks at the properties of the

critical system near a surface or near the boundaries. In the case of Hamiltonian walks or dense polymers, these surface effects will correspond to parts of the polymers located near the boundary surface. One could object rightly that a single Hamiltonian walk or circuit, in the case of free boundary conditions, always visits the surface and this should correspond to no new effect. However, when one looks at a correlation of k circuits near the surface for $k > 1$, one expects new critical exponents. For dilute polymers, we have studied[39] the surface exponents x_L^S corresponding to a vertex of L polymer lines tied near a surface. This was done with a transfer matrix study on lattice strips with free boundary conditions. In the continuum limit, this corresponds[38] to Dirichlet boundary conditions, i.e., to the *ordinary* surface transition.[37,40] The correlation function (20) generalized to two points X and Y near the surface line and L spins then has the surface critical behavior

$$G^S_{O(n),L}(X-Y) = |X-Y|^{-2x_L^S} \tag{27}$$

(Note that L is $2k$ for k polygons.) We found [39] by identification the values

$$x_L^S = h_{L+1,1} \qquad (\beta = \beta_c) \tag{28}$$

where $h_{p,q}$ is given by Kac formula (24) for[39] $n=0$, $m=2$, $C=0$ (dilute polymers). Actually, this formula, properly interpreted, holds for the surface exponents of the $O(n)$ model at the critical point β_c for $n \in [-2, 2]$. It suffices to parametrize the conformal parameter m in (24) by $(m+1)/m = g'$, where g' is the former Coulomb gas coupling constant (18a) of the $O(n)$ model for the critical point (18b). Hence, one finds from (24) and (28)

$$x_L^S = \tfrac{1}{4} g' L^2 + \tfrac{1}{2} L(g'-1) \tag{29}$$

This Coulombic formula now holds also for the other analytic determination (18c) of g', corresponding to the low-temperature phase. The associated parameter m in conformal invariance is then given by $m/(m+1) = g'$, and (29) reads

$$x_L^S = h_{1,L+1} \qquad (\beta > \beta_c) \tag{30}$$

in the dense phase of the $O(n)$ model. [This permutation of Kac indices in $h_{p,q}$—compare with (28)—is a general feature when going from the critical point β_c to the low-temperature phase $\beta > \beta_c$.]

Now, consider the correlation functions $G_k^S(X-Y)$ of k Hamiltonian circuits between two points X and Y close to the boundary of the Manhat-

tan lattice. By the same method as before, we can write these correlation functions as those of k polygons in the Potts model. The surface critical exponents associated with the Coulomb gas representation (11) of the Potts model will then be equal to the surface exponents (29) of the low-temperature phase of the $O(n)$ model by the same universality rule. Hence, their expression in the Potts model is simply

$$x_{2k}^{S} = \tfrac{1}{4} g k^2 + \tfrac{1}{4} k(g-4) \tag{31}$$

which coincides with (29) for $2k = L$ and $g' = \frac{1}{4} g$ [Eq. (19)].

Let us consider now the true Hamiltonian walks on Manhattan, i.e., the limit $z = q^{1/2} = n = 0$, with $g = 2$, $g' = 1/2$. We find from (29) or (31) the surface exponents

$$x_{2k}^{S} = \tfrac{1}{2} k(k-1) \tag{32}$$

The corresponding dense polymer surface exponents for L chains, L not necessarily even, are $x_L^S = (L/8)(L-2)$, and are studied in detail in Ref. 11 along with the various surface critical exponents derived from them. In (28), the case $L = 2$, $k = 1$ corresponds to the correlations of two surface points of a single polygon. We find from (32) that $x_2^S = 0$, as in the bulk $x_1 = x_2' = 0$. Hence, there is no difference for the points of a polygon or of a Hamiltonian circuit to be inside the lattice or at the borderline, as expected. But for $k > 1$, the nontrivial surface exponents (32) appear.

8. HAMILTONIAN CONDENSATION

On may wonder what happens above $z = n = 2$, $q = 4$, i.e., at higher fugacities for Hamiltonian walks. For $q > 4$, the Potts model has a first-order transition.[18] The $O(n)$ model for $n > 2$ has a zero critical temperature. The identification of Hamiltonian circuits with a Potts model still holds at the special point (6a), which is the critical point for $q \in [0, 4]$. For $q > 4$ it corresponds to the Potts first transition line. At this first-order transition point, the correlations are short range with an exponential decay. This corresponds to the condensation of Hamiltonian circuits into a liquid of small loops for $z > 2$, which are no longer critical. Above $q = 4$, the Potts free energy at the critical temperature T_c is given by another analytic expression,[18] different from (9) for $q \leqslant 4$, but continuously related to it by an essential singularity.

9. CONCLUSION

We have considered the particular case of Hamiltonian walks on a Manhattan lattice. Given Kasteleyn's solution for a single Hamiltonian

walk on a Manhattan torus, one could have concluded that this system is simple and too peculiar to be of general interest. We showed here that the grand canonical Manhattan walks are in fact a representation of a *critical* Potts model (with free boundary conditions). This first shows that Manhattan walks form a critical system. Moreover, since then Potts critical exponents so derived are also those of an $O(n)$ model in two dimensions below T_c, the universality is ensured. Both the Manhattan orientation and even the Hamiltonian constraint are irrelevant. The existence of these identifications was possible thanks to the peculiar geometry of the Manhattan lattice. This, of course, is not sufficient to conclude that for all lattices, Hamiltonian walks will be universal. But there is a serious hope for most of them. Further studies on other lattices would be interesting.

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